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MINIMUM-WEIGHT DESIGN OF STATICALLY DETERMINATE TRUSSES SUBJECT TO MULTIPLE CONSTRAINTS

J.-M. CHERN and W. PRAGER

Division of Engineering, Brown University, Providence, Rhode Island

Abstract—This paper is concerned with the minimum-weight design of a statically determinate elastic truss that must satisfy both stress and displacement constraints under one or more systems of loads. As used in this paper, compliance is the virtual work of specified dummy loads on the displacements that are caused by the actual loads. In particular, when a single dummy load of unit intensity is used, the compliance is the displacement of its point of application in the direction of the dummy load that is caused by the actual loads.

The stress constraints considered in the paper specify an upper bound for the absolute value of the axial stress in each bar, which need not be the same for all bars.

Sections 1 and 2, respectively, treat the cases of two compliance constraints for a single state of loading and a single compliance constraint for two alternative states of loading. In both cases, the mathematical problem is one of convex programming. To obtain physical insight that may prove useful in the discussion of more complex problems of structural optimization, both cases are treated analytically as far as possible before numerical methods are invoked.

1. TWO COMPLIANCE CONSTRAINTS FOR SINGLE STATE OF LOADING

CONSIDER a statically determinate elastic truss of given layout of bars, and assume that two compliance constraints have to be satisfied for a single system of loads. Since the truss is determinate, the bar forces S_i, S_{i1}, S_{i2} that the actual loads and the dummy loads for the two compliance constraints produce in the ith bar do not depend on the choice of the cross-sectional areas of the bars. If length, cross-sectional area, and allowable stress for the ith bar are denoted by L_i, A_i and $\hat{\sigma}_i$, the compliance constraints may be written as

$$\Sigma S_i S_{i1} L_i/A_i - C_1 \leq 0, \qquad \Sigma S_i S_{i2} L_i/A_i - C_2 \leq 0, \tag{1.1}$$

where Young's modulus, which is supposed to be the same for all bars, has been absorbed in the prescribed bounds C_1 and C_2. The stress constraint for the ith bar is

$$S_i^2/A_i^2 - \hat{\sigma}_i^2 \leq 0. \tag{1.2}$$

Since the total volume of the bars is $\Sigma L_i A_i$, we form the Lagrangian function

$$\mathscr{L} = \Sigma L_i A_i + \lambda_1(\Sigma S_i S_{i1} L_i/A_i - C_1) + \lambda_2(\Sigma S_i S_{i2} L_i/A_i - C_2) + \Sigma \gamma_i (S_i^2/A_i^2 - \hat{\sigma}_i^2), \tag{1.3}$$

where λ_1, λ_2 and the γ_i are nonnegative Lagrangian multipliers. Because the products $S_i S_{i1}$ and $S_i S_{i2}$ need not be positive, $\mathscr{L}$ is not a convex function of the cross sectional areas A_i. Note, however, that $\mathscr{L}$ is convex in the variables $\alpha_i = 1/A_i$:

$$\mathscr{L} = \Sigma L_i/\alpha_i + \lambda_1(\Sigma S_i S_{i1} L_i \alpha_i - C_1) + \lambda_2(\Sigma S_i S_{i2} L_i \alpha_i - C_2) + \Sigma \gamma_i (S_i^2 \alpha_i^2 - \hat{\sigma}_i^2). \tag{1.4}$$

The Kuhn–Tucker theorem [2] is applicable to this function and furnishes necessary and sufficient conditions for *global* optimality, which are given below in terms of the cross-sectional areas rather than their reciprocals:

$$1-\lambda_1 S_i S_{i1}/A_i^2-\lambda_2 S_i S_{i2}/A_i^2 = 2\gamma_i S_i^2/(A_i^3 L_i), \tag{1.5a}$$

$$\Sigma S_i S_{i1} L_i/A_i - C_1 \leq 0 \quad \text{if } \lambda_1 \gtreqless 0, \tag{1.5b}$$

$$\Sigma S_i S_{i2} L_i/A_i - C_2 \leq 0 \quad \text{if } \lambda_2 \gtreqless 0, \tag{1.5c}$$

$$S_i^2/A_i^2-\hat{\sigma}_i^2 \leq 0 \quad \text{if } \gamma_i \gtreqless 0. \tag{1.5d}$$

It will be convenient to use the following dimensionless variables, which are defined in terms of a reference length l and reference load intensity p, and a reference stress $\hat{\sigma}$:

$$\begin{aligned} l_i &= L_i/l, \quad a_i = A_i\hat{\sigma}/p, \quad s_i = S_i/p, \quad \sigma_i = \hat{\sigma}_i/\hat{\sigma}, \\ s_{i1} &= S_{i1}/\bar{p}, \quad s_{i2} = S_{i2}/\bar{p}, \\ c_1 &= C_1/(\bar{p}\hat{\sigma}l), \quad c_2 = C_2/(\bar{p}\hat{\sigma}l). \end{aligned} \tag{1.6}$$

In the second and third lines of (1.6), $\bar{p}$ is the unit load intensity. The optimality conditions (1.5a)–(1.5d) are converted to these dimensionless variables by replacing the capitals A, L and S by their lower case equivalents and omitting the circumflex from $\hat{\sigma}_i$ in (1.5d).

In view of (1.5d),

$$a_i \geq |s_i|/\sigma_i \equiv \hat{a}_i. \tag{1.7}$$

The bars of the truss may be divided into two groups according to whether $a_i > \hat{a}_i$ (group G^*) or $a_i = \hat{a}_i$ (group G^{**}). Note that the cross-sectional areas of the bars in groups G^* and G^{**} are respectively governed by compliance or stress constraints. It follows from (1.5d) and (1.5a) that

$$\gamma_i = 0, \quad \lambda_1 s_i s_{i1}+\lambda_2 s_i s_{i2} = a_i^2 > \hat{a}_i^2 \quad \text{for } G^*, \tag{1.8a}$$

$$\gamma_i > 0, \quad \lambda_1 s_i s_{i1}+\lambda_2 s_i s_{i2} < a_i^2 = \hat{a}_i^2 \quad \text{for } G^{**}. \tag{1.8b}$$

Optimization of the truss involves identifying the members of the two groups and determining the values of λ_1 and λ_2 for each pair of prescribed values c_1 and c_2. In view of the second equation (1.8a), the bars for which both $s_i s_{i1}$ and $s_i s_{i2}$ are nonpositive must be in group G^{**}. Members, however, for which at least one of these expressions is positive will be in G^* or G^{**} according to whether $\lambda_1 s_i s_{i1}+\lambda_2 s_i s_{i2}-\hat{a}_i^2$ is positive or negative. Thus, the two groups are completely specified when λ_1 and λ_2 are known. In the example below, we shall therefore first use an *inverse* method, whose usefulness has been pointed out by Martin [3, 4]. Whereas, in the actual problem, the compliances are prescribed and the Lagrangian multipliers are not known beforehand, this method assumes the multipliers to be known and treats the compliances as functions of the multipliers. After the general dependence of optimal design on compliances has been discussed in this manner, the solution of the original design problem can be reduced to that of a nonlinear equation.

Example

The layout of the truss and the given loads are shown in Fig. 1(a). The dummy loads P_1, P_2 for the two compliance constraints are assumed to be downward vertical loads of

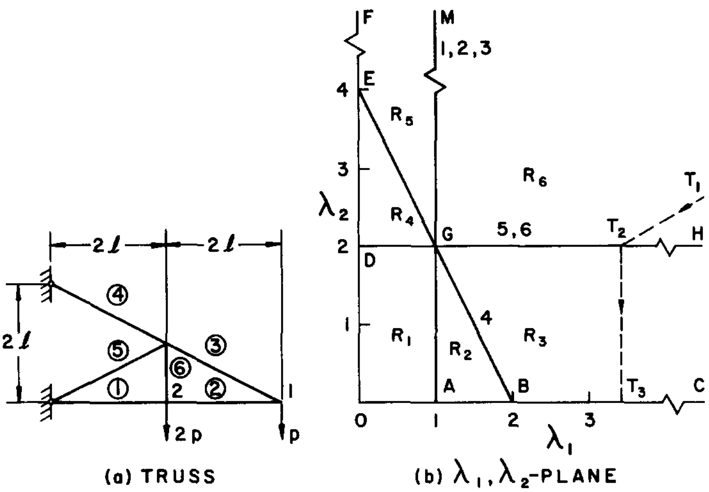

FIG. 1(a, b). Truss and λ_1, λ_2-plane for example of Section 1.

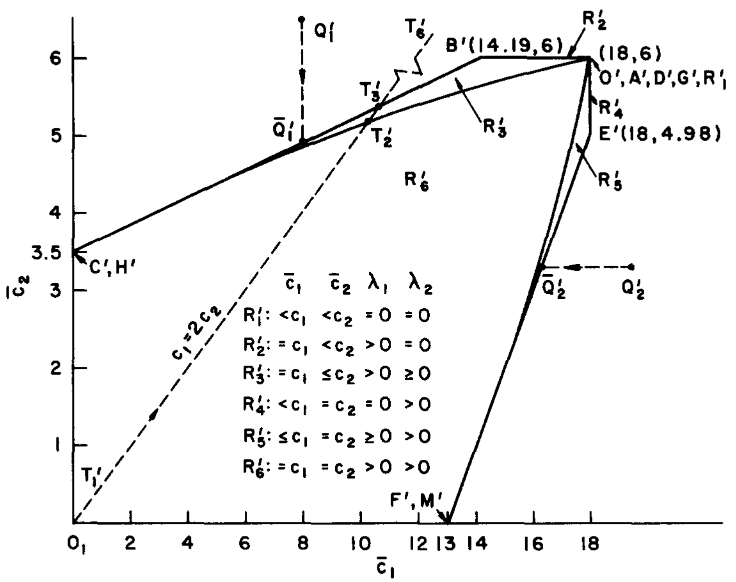

FIG. 1(c). c_1, c_2-plane for example of Section 1.

the unit intensity $\bar{p}$ that are applied to the joints 1 and 2, respectively. Thus, C_1 is the product of Young's modulus and the downward deflection of joint 1 caused by the given loads, and C_2 can be interpreted in a similar manner.

Assuming $\sigma_i = 1$ for all bars, we obtain Table 1.

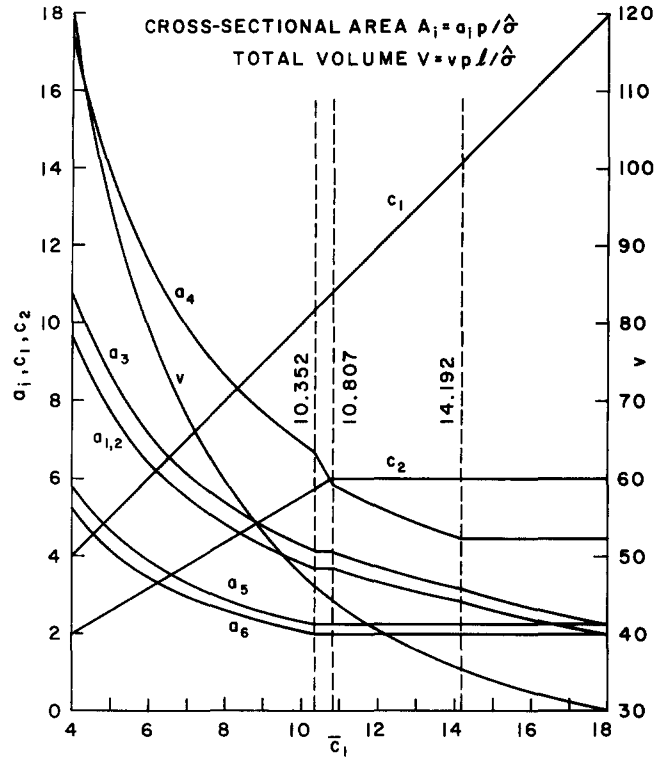

Fig. 1(d). Optimal truss of example in Section 1 with $c_1 = 2c_2$.

Table 1. Data for example truss

i	1	2	3	4	5	6
l_i	2	2	$\sqrt{5}$	$\sqrt{5}$	$\sqrt{5}$	1
s_i	-2	-2	$\sqrt{5}$	$2\sqrt{5}$	$-\sqrt{5}$	2
s_{i1}	-2	-2	$\sqrt{5}$	$\sqrt{5}$	0	0
s_{i2}	0	0	0	$\sqrt{(5)}/2$	$-\sqrt{(5)}/2$	1
$\hat{a}_i$	2	2	$\sqrt{5}$	$2\sqrt{5}$	$\sqrt{5}$	2

In the λ_1, λ_2-plane, the inequality in (1.8a) define a half-plane the points of which correspond to values of λ_1, λ_2 for which bar i belongs to group G^*. In Fig. 1(b), the boundaries of these half-planes are marked by the appropriate bar numbers written on the side of the boundary that corresponds to group G^*. The boundaries divide the λ_1, λ_2-place into six regions marked $R_1, R_2, \ldots, R_6$. For each region, the bars of groups G^* and G^{**} can be read off the figure as shown in Table 2.

Summation over the bars in groups G^* and G^{**} will be denoted by Σ^* and Σ^{**}, respectively. With

$$\bar{c}_1 = \Sigma s_i s_{i1} l_i/a_i, \qquad \bar{c}_2 = \Sigma s_i s_{i2} l_i/a_i, \tag{1.9}$$

TABLE 2. GROUPS G^* AND G^{**} FOR VARIOUS REGIONS

Region	R_1	R_2	R_3	R_4	R_5	R_6
Bars in G^*	–	1–3	1–4	5, 6	4–6	1–6
Bars in G^{**}	1–6	4–6	5, 6	1–4	1–3	–

the mapping of the λ_1, λ_2-plane onto the $\bar{c}_1, \bar{c}_2$-plane is defined by (1.8a) and (1.8b) as follows:

$$\bar{c}_j = \Sigma^* s_i s_{ij} l_i / (\lambda_1 s_i s_{i1} + \lambda_2 s_i s_{i2})^{\frac{1}{2}} + \Sigma^{**} s_i s_{ij} l_i / \hat{a}_i, \qquad (j = 1, 2). \tag{1.10}$$

The $\bar{c}_1, \bar{c}_2$-plane is shown in Fig. 1(c); point A' corresponds to point A in the λ_1, λ_2-plane, and a similar notation is used for corresponding regions. Note that a region of the λ_1, λ_2-plane may be mapped onto a region or a line segment of the $\bar{c}_1, \bar{c}_2$-plane. Consider, for example, region R_2, for which Table 2 shows that bars 1, 2, 3 are in group G^*, i.e. not fully stressed. Since $s_{12} = s_{22} = s_{32} = 0$, it follows from (1.10) that in R_2 the quantities $\bar{c}_1, \bar{c}_2$ are independent of λ_2. Thus, any point in this region may be replaced by, say, its projection on AB, and instead of considering the region ABG, we may consider the line AB, which is mapped into $A'B'$ in Fig. 1(c). The fact that λ_2 may be taken as zero in region R_2 indicates that the second compliance constraint is not relevant in R_2. Similarly, in region R_4 with bars 5, 6, in G^* and $s_{51} = s_{61} = 0$, the quantities $\bar{c}_1, \bar{c}_2$ depend only on λ_1. Finally, in R_1 with no bar in G^*, the quantities $\bar{c}_1, \bar{c}_2$ are independent of both λ_1 and λ_2. In Fig. 1(c), the regions R_2, R_4 and R_1 are mapped into the line segments R'_2, R'_4 and the point R'_1. Note that the semi-infinite positive axes in the λ_1, λ_2-plane are mapped into the lines $0'B'C'$ and $0'E'F'$ of the $\bar{c}_1, \bar{c}_2$-plane.

Using c_1, c_2-scales and -axes that are identical with the scales and axes adopted for $\bar{c}_1, \bar{c}_2$, let a pair of prescribed values c_1, c_2 of upper bounds on compliances be represented by the point Q', and let the actual compliances of the optimal truss be represented by the point $\bar{Q}'$ which must be in one of the six regions $R_1, R_2, \ldots, R_6$. In view of (1.5b, c), if $\bar{Q}'$ is on the line $C'B'0'$, we have $\bar{c}_1 = c_1, \bar{c}_2 < c_2$ since $\lambda_1 > 0, \lambda_2 = 0$. Accordingly, for Q' that is just above the line $C'B'0'$, the corresponding $\bar{Q}'$ is the vertical projection of Q' on the line $C'B'0'$; in Fig. 1(c), $\bar{Q}'_i$ corresponds to Q'_1. Similarly, since $\lambda_1 = 0$, $\lambda_2 > 0$ and $\bar{c}_1 < c_1$, $\bar{c}_2 = c_2$ if $\bar{Q}'$ is on the line $F'E'0'$, we conclude that for a Q' just to the right of the line $F'E'0'$, the corresponding $\bar{Q}'$ is the horizontal projection of $0'$ on the line $F'E'0'$. Thus, $\bar{Q}'_2$ in Fig. 1(c) corresponds to Q'_2. The point R'_1 with $\lambda_1 = \lambda_2 = 0$ and $\bar{c}_1 < c_1, \bar{c}_2 < c_2$ corresponds to all points Q' that are above and to the right of R'_1. Finally, for the point Q' that is in one of the regions R'_3, R'_5, R'_6 but not exactly on the line segments $C'B'0'$ or $F'E'0'$, we have $\bar{Q}' \equiv Q'$ since $\lambda_1 > 0, \lambda_2 > 0$ and $\bar{c}_1 = c_1, \bar{c}_2 = c_2$; if Q' is on the line segments $C'B'0'$ or $F'E'0'$, $\bar{c}_1$ or $\bar{c}_2$ is arbitrarily close to c_1 or c_2 while λ_1 or λ_2 is arbitrarily close to zero.

When the region to which the point $\bar{Q}'$ belongs is determined in this manner for the prescribed point Q', the members of the groups G^* and G^{**} are readily identified from Table 2, and the cross-sectional areas can be calculated from (1.8) after λ_1 and λ_2 have been found from (1.10). If $c_1 < \bar{c}_1$ and $c_2 < \bar{c}_2$, we have $\lambda_1 = \lambda_2 = 0$ and the optimal truss is fully stressed with $a_i = \hat{a}_i$ for all bars. If $c_1 < \bar{c}_1$ or $c_2 < \bar{c}_2$, we have $\lambda_1 = 0$ or $\lambda_2 = 0$, and the nonvanishing λ_2 or λ_1 can be directly calculated from (1.10) with $j = 2$ or $j = 1$.

If $c_1 = \bar{c}_1, c_2 = \bar{c}_2$, then $\lambda_1 > 0, \lambda_2 > 0$ and we may set $\lambda_2/\lambda_1 = y$ and obtain the following equations from (1.10):

$$\sqrt{\lambda_1} = \{\Sigma^* s_i s_{i1} l_i/(s_i s_{i1} + y s_i s_{i2})^{\frac{1}{2}}\}/\{c_1 - \Sigma^{**} s_i s_{i1} l_i/\hat{a}_i\}$$
$$= \{\Sigma^* s_i s_{i2} l_i/(s_i s_{i1} + y s_i s_{i2})^{\frac{1}{2}}\}/\{c_2 - \Sigma^{**} s_i s_{i2} l_i/\hat{a}_i\}. \tag{1.11}$$

After y has been found from the second equation (1.11), λ_1 can readily be evaluated from the first equation (1.11) and λ_2 is then given by $\lambda_2 = y\lambda_1$.

As a numerical example, we treat the case where the prescribed positive values c_1, c_2 satisfy the equation $c_1 = 2c_2$, which is represented by the semi-infinite line $T'_1T'_6$ in Fig. 1(c). Let T'_4 and T'_5 [not shown in Fig. 1(c)] be the points on this line with the same abscissas as B' and $0'$ respectively. The $\bar{c}_1, \bar{c}_2$-values of the corresponding optimal trusses are on the line segments $T'_1T'_2$, $T'_2T'_3$, T'_3B' and $B'0'$ which are respectively in the regions R'_6, R'_3, R'_3 with $\lambda_2 = 0$, and R'_2. After the members of groups G^* and G^{**} have been determined from Table 2, the λ_1, λ_2-values for given c_1 are calculated from (1.10) or (1.11) [line segments T_1T_2, T_2T_3, T_3B and BA in Fig. 1(b)]. We note that for the c_1, c_2-values on the line segment $T'_1T'_2$ of Fig. 1(c), we have no bar in G^{**} and $y = 0{\cdot}5799$ as calculated from the second equation (1.11). Accordingly, the λ_1, λ_2-values are on the line segment T_1T_2 of Fig. 1(b). The fact that $\lambda_2 = 2$ at T_2 yields $c_1 = 10{\cdot}352$ at T'_2. For the line segment $T'_2T'_3$, Tables 1, 2 and the first equation (1.11) with $c_1 = 2c_2$ show that $\lambda_1 = 3{\cdot}449$ independently of c_1. In Fig. 1(b), the corresponding λ_1, λ_2-values thus are on the vertical line T_2T_3. The fact that $y = 0$ at T_3 furnishes $c_1 = 10{\cdot}807$ at T'_3. These optimal designs are indicated in Fig. 1(d) in dependence on $\bar{c}_1$. In addition, the total volume $v = \Sigma a_i l_i$ is also shown in this figure. We note that $\bar{c}_1 = c_1$ for $0 < c_1 \leq 18$, $\bar{c}_1 = 18$ for $c_1 \geq 18$, $c_2 = \bar{c}_2 = c_1/2$ for $0 < c_1 \leq 10{\cdot}807$ and $\bar{c}_2 = 6$ for $c_1 \geq 10{\cdot}807$.

2. ONE COMPLIANCE CONSTRAINT FOR TWO ALTERNATIVE STATES OF LOADING

Consider again a statically determinate elastic truss of given layout of bars, and assume that a compliance constraint associated with a given set of dummy loads has to be satisfied for each of two alternative systems of loads. As in Section 1, the bar forces $S_{i1}, S_{i2}, \bar{S}_i$ that the actual and dummy loads cause in the ith bar are independent of the choice of the cross-section areas of bars. If we again denote the length, cross-sectional area and allowable stress for the ith bar by L_i, A_i and $\hat{\sigma}_i$, the compliance constraints may be written as

$$\Sigma S_{i1}\bar{S}_i L_i/A_i - C_1 \leq 0, \qquad \Sigma S_{i2}\bar{S}_i L_i/A_i - C_2 \leq 0, \tag{2.1}$$

and the stress-constraints for the ith bar as

$$S_{i1}^2/A_i^2 - \hat{\sigma}_i^2 \leq 0, \qquad S_{i2}^2/A_i^2 - \hat{\sigma}_i^2 \leq 0. \tag{2.2}$$

In (2.1), Young's modulus has been absorbed in the prescribed bounds C_1 and C_2.

If we introduce the nonnegative Lagrangian multipliers $\lambda_1, \lambda_2, \gamma_{i1}, \gamma_{i2}$ and form the Lagrangian function

$$\mathscr{L} = \Sigma L_i A_i + \lambda_1(\Sigma S_{i1}\bar{S}_i L_i/A_i - C_1) + \lambda_2(\Sigma S_{i2}\bar{S}_i L_i/A_i - C_2)$$
$$+ \Sigma\gamma_{i1}(S_{i1}^2/A_i^2 - \hat{\sigma}_i^2) + \Sigma\gamma_{i2}(S_{i2}^2/A_i^2 - \hat{\sigma}_i^2), \tag{2.3}$$

we may follow a procedure similar to that of Section 1 and obtain the optimality conditions:

$$1-\lambda_1 S_{i1}\bar{S}_i/A_i^2-\lambda_2 S_{i2}\bar{S}_i/A_i^2 = 2(\gamma_{i1}S_{i1}^2+\gamma_{i2}S_{i2}^2)/(A_i^3 L_i) \quad (2.4a)$$

$$\Sigma S_{i1}\bar{S}_i L_i/A_i - C_1 \le 0 \quad \text{if } \lambda_1 \gtreqless 0, \quad (2.4b)$$

$$\Sigma S_{i2}\bar{S}_i L_i/A_i - C_2 \le 0 \quad \text{if } \lambda_2 \gtreqless 0, \quad (2.4c)$$

$$S_{i1}^2/A_i^2-\hat{\sigma}_i^2 \le 0 \quad \text{if } \gamma_{i1} \gtreqless 0, \quad (2.4d)$$

$$S_{i2}^2/A_i^2-\hat{\sigma}_i^2 \le 0 \quad \text{if } \gamma_{i2} \gtreqless 0. \quad (2.4e)$$

Let the representative load intensity of the first loading system be denoted by p_1 and that of the second loading system by $p_2 \equiv rp_1$. In the following we shall use the dimensionless variables:

$$l_i = L_i/l, \quad a_i = A_i\hat{\sigma}/p_1, \quad s_{i1} = S_{i1}/p_1, \quad s_{i2} = S_{i2}/(rp_1), \quad \sigma_i = \hat{\sigma}_i/\hat{\sigma},$$
$$\bar{s}_i = \bar{S}_i/\bar{p}, \quad c_1 = C_1/(\bar{p}\hat{\sigma}l), \quad c_2 = C_2/(r\bar{p}\hat{\sigma}l), \quad (2.5)$$

where $l, p_1, \hat{\sigma}$ are reference length, reference load intensity and reference stress, and $\bar{p}$ denotes the unit load intensity. We may then, as in Section 1, convert the optimality conditions (2.4a)–(2.4e) to these dimensionless variables.

In view of (2.4d, e),

$$a_i \ge \max\{|s_{i1}|, r|s_{i2}|\}/\sigma_i \equiv \hat{a}_i. \quad (2.6)$$

The bars of the truss may again be divided into two groups according to whether $a_i > \hat{a}_i$ (group G^*) or $a_i = \hat{a}_i$ (group G^{**}). It then follows from (2.4d, e) and (2.4a) that

$$\gamma_{i1} = \gamma_{i2} = 0, \quad \lambda_1 s_{i1}\bar{s}_i+\lambda_2 s_{i2}\bar{s}_i = a_i^2 > \hat{a}_i^2 \quad \text{for } G^*, \quad (2.7a)$$

$$\gamma_{i1} > 0 \quad \text{or} \quad \gamma_{i2} > 0, \quad \lambda_1 s_{i1}\bar{s}_i+\lambda_2 s_{i2}\bar{s}_i < a_i^2 = \hat{a}_i^2 \quad \text{for } G^{**}. \quad (2.7b)$$

Optimization of the truss involves identifying the members of the two groups and determining the values of λ_1 and λ_2 for each pair of prescribed values c_1 and c_2. In view of the second equation (2.7a), the bars for which both $s_{i1}\bar{s}_i$ and $s_{i1}\bar{s}_i$ are negative must be in groups G^{**}. Members, however, for which at least one of these expressions is positive will be in G^* or G^{**} according to whether $\lambda_1 s_{i1}\bar{s}_i+\lambda_2 s_{i2}\bar{s}_i-\hat{a}_i^2$ is positive or negative. Thus, the groups are completely specified when λ_1 and λ_2 are known. The procedure of furnishing an optimal design is exactly the same as in Section 1 as shown in the example below.

Example

The layout of the truss and the given alternative loading systems of load intensity p_1, p_2 are shown in Fig. 2(a). The dummy load for compliance constraints is assumed to be a downward vertical load of unit intensity $\bar{p}$ that is applied to joint 1. Thus, C_1 and C_2 are respectively the products of Young's modulus and the downward deflections of joint 1 caused by the first and the second loading systems.

Assuming $\sigma_i = 1$ for all bars and $p_1 = p_2$, we obtain Table 3.

In the λ_1, λ_2-plane, the inequality in (2.7a) defines a half-plane the points of which correspond to values of λ_1, λ_2 for which bar i belongs to group G^*. In Fig. 2(b), the boundaries of these half-planes are marked by the appropriate bar numbers written on the side of the

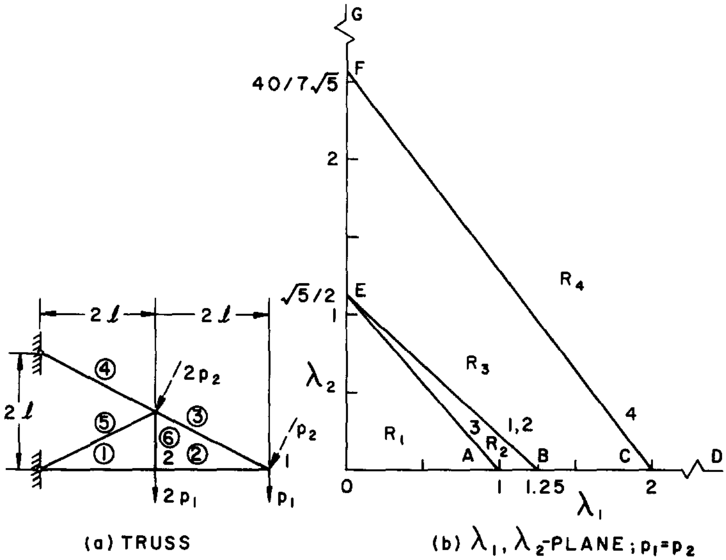

FIG. 2(a, b). Truss and λ_1, λ_2-plane for example of Section 2.

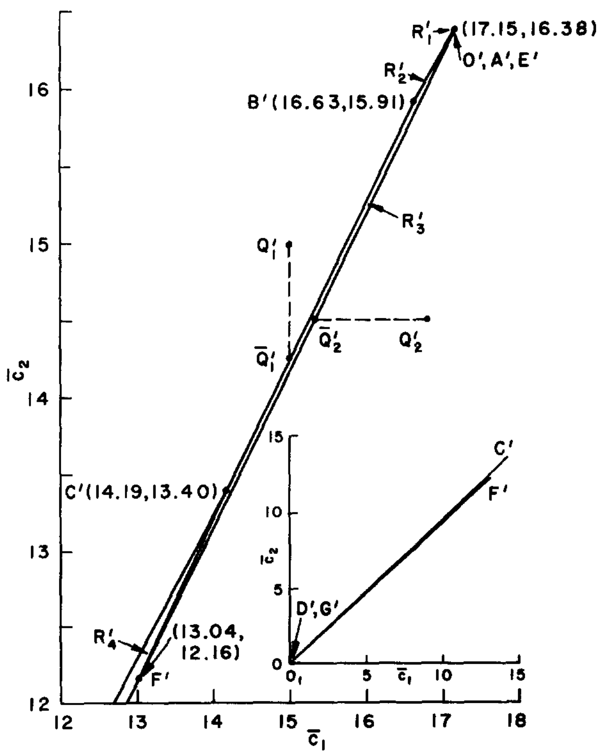

FIG. 2(c). c_1, c_2-plane for example of Section 2; $p_1 = p_2$.

TABLE 3. DATA FOR EXAMPLE TRUSS

i	1	2	3	4	5	6
l_i	2	2	$\sqrt{5}$	$\sqrt{5}$	$\sqrt{5}$	1
s_{i1}	-2	-2	$\sqrt{5}$	$2\sqrt{5}$	$-\sqrt{5}$	2
s_{i2}	$-\sqrt{5}$	$-\sqrt{5}$	2	7/2	$-5/2$	0
$\bar{s}_i$	-2	-2	$\sqrt{5}$	$\sqrt{5}$	0	0
$\hat{a}_i$	$\sqrt{5}$	$\sqrt{5}$	$\sqrt{5}$	$2\sqrt{5}$	5/2	2

boundaries that correspond to group G^*; the bars 5, 6 with $s_{i1}\bar{s}_i = s_{i2}\bar{s}_i = 0$ can never be in group G^*. The boundaries divide the λ_1, λ_2-plane into four regions marked by $R_1, R_2 \ldots, R_4$. For each region, the bars of groups G^* and G^{**} can be read off the figure as shown in Table 4.

TABLE 4. GROUPS G^* AND G^{**} FOR VARIOUS REGIONS

Region	R_1	R_2	R_3	R_4
Bars in G^*	–	3	4	1–4
Bars in G^{**}	1–6	1, 2, 4–6	1–3, 5, 6	5, 6

Summation over the bars in group G^* and G^{**} will again be denoted by Σ^* and Σ^{**}, respectively. With

$$\bar{c}_1 = \Sigma s_{i1}\bar{s}_i l_i/a_i, \qquad \bar{c}_2 = \Sigma s_{i2}\bar{s}_i l_i/a_i, \tag{2.8}$$

the mapping of the λ_1, λ_2-plane into the $\bar{c}_1, \bar{c}_2$-plane is defined by

$$\bar{c}_j = \Sigma^* s_{ij}\bar{s}_i l_i/(\lambda_1 s_{i1}\bar{s}_i + \lambda_2 s_{i2}\bar{s}_i)^{\frac{1}{2}} + \Sigma^{**} s_{ij}\bar{s}_i l_i/\hat{a}_i, \qquad (j = 1, 2). \tag{2.9}$$

The $\bar{c}_1, \bar{c}_2$-plane is shown in Fig. 2(c) where notations similar to those of the example in Section 1 are used; for instance, point A' and point R_1' correspond to point A and region R_1, respectively. The semi-infinite positive λ_1-axis in the λ_1, λ_2-plane is mapped onto the line segments $0'B'C'D'$ and the λ_2-axis onto $0'F'G'$. [Note that D' and G' are shown on insert of Fig. 2(c).] For the region R_2 with only bar 3 in G^*, from (2.9) we have

$$\bar{c}_i = K_1 + 5\sqrt{(5)}/(5\lambda_1 + 2\sqrt{(5)}\lambda_2)^{\frac{1}{2}}, \qquad \bar{c}_2 = K_2 + 5\sqrt{(5)}/(5\lambda_1 + 2\sqrt{(5)}\lambda_2)^{\frac{1}{2}}$$

where K_1 and K_2 denote the constants of the second summations in (2.9). Accordingly, the mapping depends only on the values of $5\lambda_1 + 2\sqrt{(5)}\lambda_2$. For a point M on the line segment AB of the λ_1, λ_2-plane with the abscissa $\lambda_1^{(M)}$, all points of the line EM with the equation $5\lambda_1 + 2\sqrt{(5)}\lambda_2 = 5\lambda_1^{(M)}$ are mapped onto the same point in the $\bar{c}_1, \bar{c}_2$-plane as the point M. Accordingly, the region R_2 is mapped onto the line segment $A'B'$ in the $\bar{c}_1, \bar{c}_2$-plane.

Let the prescribed values c_1, c_2 be represented again by a point Q' in the $\bar{c}_1, \bar{c}_2$-plane, and the $\bar{c}_1, \bar{c}_2$-values of the corresponding optimal truss by a point $\bar{Q}'$. Following the similar argument as in the example of Section 1, we conclude that $\bar{Q}' \equiv Q'$ if Q' is in one of the four regions $R_1' \ldots, R_4'$. If Q' is just above the line segments $D'C'B'0'$, then $\lambda_1 > 0$, $\lambda_2 = 0$, $\bar{c}_1 = c_1$, $\bar{c}_2 < c_2$, and the corresponding $\bar{Q}'$ is the vertical projection of Q' on $D'C'B'0'$; for instance, in Fig. 2(c), $\bar{Q}_1'$ corresponds to Q_1'. Similarly, if Q' is just to the right of the line segment $G'F'0'$, then $\lambda_1 = 0$, $\lambda_2 > 0$, $\bar{c}_1 < c_1$, $\bar{c}_2 = c_2$, and the corresponding $\bar{Q}'$ is the horizontal projection of Q' on $G'F'0'$; for instance, in Fig. 2(c), $\bar{Q}'_2$ corresponds to Q'_2. Finally if Q' is above and to the right of R'_1, then $\bar{Q}' \equiv R'_1$ since $\lambda_1 = \lambda_2 = 0$ and $\bar{c}_1 < c_1$, $\bar{c}_2 < c_2$.

When the region to which the point $\bar{Q}'$ belongs is indicated in this manner for prescribed Q', the members of the groups G^* and G^{**} are readily identified from Table 4. The values λ_1, λ_2 and then the cross-sectional area a_i of the optimal truss can be calculated from (2.9) and (2.7) in exactly the same manner as in Section 1.

3. CONCLUDING REMARK

In the form in which it has been presented, the method of structural optimization of trusses discussed above applies only to statically determinate trusses. Martin [4] has however been successful in applying a modification of the method to statically indeterminate beams. A similar modification for use with statically indeterminate trusses appears feasible and will form the subject of a follow-up paper.

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Абстракт—В работе обсуждается расчет на минимум веса статически определенной упругой фермы, которая должна удовлетворять огапичечичо связеа так напряжений, как и деформаций, под влиянием одной системы и более систем нагрузки. Используется в работе уступчивость являющеяся виртуальной работой специфических фиктивных нагрузок на перемещениях, вызванных действительной нагрузкой. В качестве применения единичной фиктивной нагрузки, такая уступчивость представляет перемещение точки ее приложения по направлению фиктивной нагрузки, вызванно действительными нагрузками.

Ограничения напряжений, рассматриваемые в работе, определяют верхний предел абсолютной величины осевого напряжения в каждом стержне, которое не является одинаковым для всех стержней.

Части 1 и 2, соответственно, определяют случаи двух уступчивых связей для простого состояния нагрузки и одинарной уступчивой связи двух альтернативных состояний нагрузки. Для этих двух случаев математиуеская проблема оказывается одной из задач выпуклого программирования. Оба случая описываются аналитически так далеко, как это является возможным, прежде чем будут использованы численные расчеты, чтобы в результате получить физический взгляд, который может оказаться полезным при обсуждений более сложных задач оптимализации конструкции.